An Overview of Multiplying Gaussians and Sampling

Let $y = e^{-z^2}$ then $I = \int_{-\infty}^{\infty} e^{-z^2} dz = \sqrt{\pi}$

Now let $z = \frac{x-\mu}{\sigma}$ then $dz = \frac{dx}{\sigma}$ and we have $I = \frac{1}{\sigma}\int_{-\infty}^{\infty} e^{-\left(\frac{x-\mu}{\sigma}\right)^2} dx = \sqrt{\pi}$

It is then simple to see that $\frac{1}{\sigma\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\left(\frac{x-\mu}{\sigma}\right)^2} dx = 1$ which is the gaussian distribution.

For some reason it is usual to use a slightly different form $\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = 1$

which is equivalent as we simply let $z = \frac{w}{\sqrt{2}}$ and then $I = \frac{1}{\sqrt{2}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}w^2} dw = \sqrt{\pi}$

The gaussian used below is therefore: $g(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

Product of Gaussians

To avoid the use of confusing subscripts we define the two gaussians as:

$g(x) = \frac{A}{a\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\alpha}{a}\right)^2}$ $h(x) = \frac{B}{b\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\beta}{b}\right)^2}$ and $f(x) = g(x).h(x)$

$f(x) = \frac{AB}{ab2\pi} e^{-\frac{1}{2}p(x)}$ where $p(x) = \left(\frac{x-\alpha}{a}\right)^2 + \left(\frac{x-\beta}{b}\right)^2$

Expanding *p(x)* we have $p(x) = \frac{1}{a^2}\left(x^2 - 2\alpha x + \alpha^2\right) + \frac{1}{b^2}\left(x^2 - 2\beta x + \beta^2\right)$

$$p(x) = \frac{b^2\left(x^2 - 2\alpha x + \alpha^2\right) + a^2\left(x^2 - 2\beta x + \beta^2\right)}{a^2b^2} = \frac{\left(a^2+b^2\right)x^2 - 2\left(a^2\beta + b^2\alpha\right)x + \left(a^2\beta^2 + b^2\alpha^2\right)}{a^2b^2}$$

$$p(x) = \frac{x^2 - 2\frac{a^2\beta + b^2\alpha}{a^2+b^2}x + \frac{a^2\beta^2 + b^2\alpha^2}{a^2+b^2}}{\left(\frac{a^2b^2}{a^2+b^2}\right)}$$

then using a well known phrase or saying… $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ (note: a,b,c not same)

we have $\frac{\left(a^2\beta + b^2\alpha\right) \pm \sqrt{\left(a^2\beta + b^2\alpha\right)^2 - \left(a^2+b^2\right)\left(a^2\beta^2 + b^2\alpha^2\right)}}{\left(a^2+b^2\right)}$

$$\sqrt{\ } = \sqrt{a^4\beta^2 + 2a^2b^2\alpha\beta + b^4\alpha^2 - a^4\beta^2 - a^2b^2\alpha^2 - a^2b^2\beta^2 - b^4\alpha^2}$$
$$\sqrt{\ } = ab\sqrt{2\alpha\beta - \alpha^2 - \beta^2} = ab(\alpha - \beta)\sqrt{-1}$$

and so we have

$$\frac{\left(a^2\beta + b^2\alpha\right) \pm ab(\alpha-\beta)\sqrt{-1}}{a^2+b^2}$$

Now, as we can always write: $\left(x - (u+v)\right)\left(x - (u-v)\right) = (x-u)^2 - v^2$ we have

$$p(x) = \frac{\left(x - \frac{a^2\beta + b^2\alpha}{a^2+b^2}\right)^2 + \left(\frac{ab(\alpha-\beta)}{a^2+b^2}\right)^2}{\left(\frac{a^2b^2}{a^2+b^2}\right)} = \left(\frac{x-\mu}{\sigma}\right)^2 + k$$

where $\sigma^2 = \frac{a^2b^2}{a^2+b^2}$ or equivalently $\frac{1}{\sigma^2} = \frac{1}{a^2} + \frac{1}{b^2}$

and $\mu = \frac{a^2\beta + b^2\alpha}{a^2+b^2}$ and $k = \frac{(\alpha-\beta)^2}{a^2+b^2} = \left(\frac{\alpha-\beta}{ab/\sigma}\right)^2$

so we have $f(x) = \frac{AB}{ab2\pi} e^{-\frac{1}{2}\left(\frac{\alpha-\beta}{ab/\sigma}\right)^2} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

The area under this curve will be $I = AB\frac{\sigma}{ab\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\alpha-\beta}{ab/\sigma}\right)^2}$

Notice that if $\alpha = \beta$ and $\sigma^2 = a^2b^2 2\pi = \frac{a^2b^2}{a^2+b^2}$ that is $a^2 + b^2 = \frac{1}{2\pi}$ then product is still normal.

Sequence of gaussians of unit hieght

Consider the gaussian $\Omega(x) = \frac{W}{w\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\omega}{w}\right)^2}$ which has integral sum wrt to *x*: $I = W$

At $x = \omega$ we require $\Omega(x) = 1$ so we have $\frac{W}{w\sqrt{2\pi}} = 1$ or $W = \sqrt{2\pi}.w$

This can be regarded as following a linear path through (W,w) space. However, the consequence is that we can consider:

$\Omega_0(x) = e^{-\frac{1}{2}\left(\frac{x-\omega}{w}\right)^2}$ which is just the basic gaussion form considered in the introduction. It has the following properties:

1. $\Omega_0(x) = 1$ at $x = \omega$
2. As $w \to \infty$ then $\Omega_0(x) = 1$ for $\forall x$
3. The integral $I = \sqrt{2\pi}.w$

Sampling with a gaussian

Let the data gaussian be $\Delta(x) = \frac{D}{d\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\delta}{d}\right)^2}$

Let the weighting gaussian be $\Omega_0(x) = e^{-\frac{1}{2}\left(\frac{x-\omega}{w}\right)^2}$

Let the product $\Phi(x) = \Delta(x).\Omega_0(x) = \frac{D}{d\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\delta-\omega}{dw/\sigma}\right)^2} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

where $\frac{1}{\sigma^2} = \frac{1}{d^2} + \frac{1}{w^2}$ and $\mu = \frac{d^2\omega + w^2\delta}{d^2 + w^2}$

This has an area $I = D\frac{\sigma}{d} e^{-\frac{1}{2}\left(\frac{\delta-\omega}{dw/\sigma}\right)^2}$

So we have $\Phi(x) = \frac{I}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

By rearranging the expression for $\mu = \frac{d^2\omega}{d^2 + w^2} + \frac{\delta}{\frac{d^2}{w^2} + 1}$ we can see that:

as $w \to \infty$ then $\mu \to \delta$, it is also clear that as $w \to \infty$ then $\sigma \to d$

So in the limit we have $\Phi(x) = \frac{D}{d\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\delta}{d}\right)^2}$ which was our original gaussian.

Degenerate Cases

Assume that the data to be sampled is a single point x with a frequency of N,

This can be written as $g(x) = \begin{cases} N : x = \delta \\ 0 : x \neq \delta \end{cases}$ or

$$\Omega_0(x) = e^{-\frac{1}{2}\left(\frac{x-\omega}{w}\right)^2}$$

Minimum Sample Size

If we have a minimum sample size then we have:

$$S_{\min} = D\frac{\sigma}{d} e^{-\frac{1}{2}\left(\frac{\delta-\omega}{dw/\sigma}\right)^2} = D\sqrt{\frac{\frac{d^2w^2}{d^2+w^2}}{d^2}} e^{-\frac{1}{2}\frac{(\delta-\omega)^2}{d^2+w^2}} = D\sqrt{\frac{w^2}{d^2+w^2}} e^{-\frac{1}{2}\frac{(\delta-\omega)^2}{d^2+w^2}}$$

$$\ln(S_{\min}) = \ln(D) + \frac{1}{2}\ln\left(\frac{w^2}{d^2+w^2}\right) - \frac{1}{2}\frac{(\delta-\omega)^2}{d^2+w^2}$$

$$2\left(\ln(S_{\min}) - \ln(D)\right) = \ln\left(\frac{w^2}{d^2+w^2}\right) - \frac{(\delta-\omega)^2}{d^2+w^2} = 2\ln(w) - \ln\left(d^2+w^2\right) - \frac{(\delta-\omega)^2}{d^2+w^2}$$